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Four-Dimensional Twisted Group Lattices

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Abstract

Four-dimensional twisted group lattices are used as models for space-time structure. Compared to other attempts at space-time deformation, they have two main advantages: They have a physical interpretation and there is no difficulty in putting field theories on these structures. We present and discuss ordinary and gauge theories on twisted group lattices. We solve the free field theory case by finding all the irreducible representations. The non-abelian gauge theory on the two-dimensional twisted group lattice is also solved. On twisted group lattices, continuous space-time translational and rotational symmetries are replaced by discrete counterparts. We discuss these symmetries in detail. Four-dimensional twisted group lattices can also be used as models for non-trivial discrete compactifications of certain ten-dimensional spaces.

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1 Introduction

Recently, there has been interest in attempts to deform the structure of space-time. Normally, space-time coordinates $x_1, x_2, \ldots, x_d$ commute: $x_j x_i = x_i x_j$. One can q-deform this algebra by assuming a non-commutative structure. For example, the quantum hyperspaces [1, 2] are defined by the algebra $x_j x_i = q_{ij} x_i x_j$, for $i < j$, where the q_{ij} are complex numbers. In the $q_{ij} \to 1$ limit, one recovers ordinary space-time.[1]

It is not completely clear what a quantum hyperspace physically represents. It appears to possess some kind of lattice structure. Because of this lack of understanding, field theories cannot yet be defined on quantum hyperspaces. For this reason, contact with standard physics is not yet possible.

In this work, we shall consider a particular deformation of space-time structure which avoids the disadvantages of quantum hyperspaces. We call these spaces twisted group lattices. They are closely related to quantum hyperspaces but have several physical interpretations. Furthermore, one can define standard field theories on twisted group lattices. Twisted group lattices are, in some sense, the natural extension of quantum hyperspaces. In this work, we focus on the four-dimensional twisted group lattice and field-theoretic aspects.

The twisted group lattice is a lattice. In general, a lattice is determined by specifying sites and bonds. By definition, two sites are nearest neighbors if there is a bond between them.

A group lattice [3, 4] is built from a discrete group G. One declares a one-to-one correspondence between the elements of G and the sites of the lattice. Hence the number of sites in the lattice is the order of the group $o(G)$, i. e., the total number of elements of the group. By convention, the origin corresponds to the identity element e of G. The nearest neighbor sites are determined by a subset NN of elements of G, satisfying the property that if $h \in NN$ then $h^{-1} \in NN$. Usually NN is not chosen to be a subgroup of G. A site g' is a nearest neighbor site of g if $g' g^{-1} \in NN$. The nearest neighbor sites of the origin e are the elements in NN. In general, the nearest neighbors of g are hg for $h \in NN$. Since the bonds $\mathcal{B}$ of the lattice are lines between two nearest neighbor sites: $\mathcal{B} = \{[hg, g]$, such that $g \in G, h \in NN\}$. Hence, a group lattice is not only determined by G but also by NN. For more details on group lattices, see Refs. [3]- [6].

[1]One can consider more general relations, such as $x_j x_i = R_{ji}{}^{kl} x_k x_l$, where $R_{ji}{}^{kl}$ satisfy the Yang-Baxter equation. Such spaces are even more difficult to interpret.

For complicated *NN* and G, the corresponding group lattice can appear quite complex. However, a group lattice has more structure than a random lattice. The structure is governed by G. Group lattices are homogeneous in the sense that the lattice appears the same when viewed from any site.

One can consider the free propagation of bosons or fermions on a group lattice. Such particles are allowed to hop from site to site. Hopping parameters control the ease or difficulty of passing over a bond. One reason that group lattices are interesting is that the corresponding statistical mechanics system can be solved exactly if the irreducible representations of G are known. The solution is given in Ref. [3]. The solution method uses the group analog of the Fourier transform. Furthermore, if interactions are included then a perturbative expansion of the theory is possible because propagators exist and can be computed. Thus, it is straightforward to put field theories on group lattices and perform calculations.

Because the concept of a group lattice is general, there should be many applications. Indeed, the carbon atoms of the buckyball C_{60} sit at the sites of a group lattice based on the group A_5, the alternating group on 5 elements. Using this result and some approximations, one can compute the electronic structure of C_{60}. The agreement of the group-lattice results with experiments is good [7, 8].

The regular periodic hyper-cubic lattices in d-dimensions of size $L_1 \times L_2 \times \ldots \times L_d$ correspond to a group lattice based on an abelian group. The group is $G = Z_{L_1} \times Z_{L_2} \times \ldots \times Z_{L_d}$. Let x_i be the generator of the ith abelian factor, Z_{L_i}, where $x_i^{L_i} = e$. Let *NN* consist of the x_i and their inverses, i. e., $NN = \{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_d, x_d^{-1}\}$. Each $g \in G$ is of the form $g = x_1^{n_i} x_2^{n_2} \ldots x_d^{n_d}$ with $0 \leq n_i \leq L_i - 1$. Associate the point $(n_1, n_2, \ldots, n_d)$ in the d-dimensional hyper-cubic lattice with this g. Since the nearest neighbors to $x_1^{n_i} x_2^{n_2} \ldots x_d^{n_d}$ are obtained by multiplying by an x_i or an x_i^{-1}, the nearest neighbors of $(n_1, n_2, \ldots, n_d)$ are $(n_1, n_2, \ldots, n_i \pm 1, \ldots, n_d)$. Clearly, one obtains the hyper-cubic periodic d-dimensional lattice. The infinite hyper-cubic lattice is achieved by replacing the Z_{L_i} by the group of integers Z. This can be achieved by taking the $L_i \to \infty$ and can be regarded as a thermodynamic limit.

The generator x_i can be thought of as taking a step in the ith direction. Likewise x_i^{-1} is a step in the negative i-direction. The element $g = x_1^{n_i} x_2^{n_2} \ldots x_d^{n_d}$ is obtained by starting at the origin and taking n_1 steps in the 1-direction, followed by n_2 steps in the 2-direction, …, and finally, x_d steps in the d-direction. Because the group is abelian, it doesn't matter in which order one proceeds.

The d-dimensional twisted group lattice is based on the ordinary d-dimensional

hyper-cubic lattice. One uses the above notion that x_i and x_i^{-1} correspond to steps in the plus or minus i-direction. However, paths which go around closed loops do not necessarily return to the origin e. Instead one arrives at new group elements. This introduces local twisting into the system. Let N_{ij}, for $i < j$, be a set of fixed positive integers. When going around an elementary plaquette in the i-j plane, one arrives at a new element z_{ij} which satisfies $z_{ij}^{N_{ij}} = e$. In other words, $x_j^{-1}x_i^{-1}x_jx_i = z_{ij}$, for $i < j$.[2] The z_{ij} commute with all elements of the group. In moving around an elementary plaquette in the i-j plane, one must go around it N_{ij} times to return to the origin. A general path returns to the origin if and only if the region projected onto each of the i-j planes has an area which is 0 (mod N_{ij}). The precise definition of G is

$$\begin{aligned} G \; = \; \Big\{ & x_1^{n_1}x_2^{n_2}\ldots x_d^{n_d}\prod_{i<j} z_{ij}^{n_{ij}} \;, \text{ such that } n_i = 0, 1, \ldots, L_i - 1, \\ & n_{ij} = 0, 1, \ldots, N_{ij} - 1,\; x_jx_ix_j^{-1}x_i^{-1} = z_{ij} \text{ for } i < j,\; x_kz_{ij} = z_{ij}x_k, \\ & z_{ij}z_{kl} = z_{kl}z_{ij},\; x_i^{L_i} = z_{ij}^{N_{ij}} = e,\; N_{ij} \text{ divides } L_i \text{ and } L_j \text{ for all } i \text{ and } j \Big\} \;. \end{aligned} \tag{1.1}$$

Part of the interest in these twisted group lattices lies in the connection with quantum hyperspaces [1, 2]. Indeed, if the z_{ij} are represented by complex numbers, one obtains the quantum hyperspaces, of which the $d = 2$ case is the quantum plane [1]. Different values of the $\{N_{ij}\}$ and the $\{L_i\}$ produce different twisted lattices. The irreducible representations for the general two- and three-dimensional twisted group lattices were found in Refs. [5, 6]. Hence, the propagation of free particles on those group lattices has been exactly solved. Propagation on twisted lattices is very different from regular lattices because closed paths are obtained only if the vanishing (mod N_{ij}) area constraints are satisfied.

In this paper, we solve the four-dimensional twisted group lattice. Some motivation for studying this case is that these lattices might be useful in connection with four-dimensional physics. For example, one can consider field theories on group lattices. An open question is whether such field theories play a role in particle physics, perhaps approximately or in a limit. Other motivation comes from quantum groups and quantum hyperspaces. Because of the close similarity between the twisted group lattices and quantum hyperspaces, one might be able to gain insight into the latter.

The solution method requires finding the irreducible representations of the four-

[2]Our convention is that the elements associated with movements are multiplied from right to left. For example, x_jx_i represents first a movement in the i-direction and then a movement in the j-direction.

dimensional twisted group lattices. Some elementary number theory plays a role. In Sect. 2, we perform a prime factorization of the system. This allows one to solve the general problem by analyzing smaller subsectors. In Sect. 3, we obtain all the irreducible representations. This allows us to solve the free theory case. Since we are interested in possible physical applications, we consider field theories on generic group lattices in Sect. 4. In particular, non-abelian gauge theories can be defined using methods similar to those of Wilson [9]. The gauge theory on the two-dimensional twisted lattice can be solved exactly. The solution is presented in Sect. 4.2. Another topic concerns the gamma matrix structure which arises for the $N_{ij} = 2$ case in a particular subsector. We find that Kogut-Susskind staggered fermions naturally arise. In Sect. 4.4, the space-time discrete symmetries of the twisted group lattices are analyzed. At this early stage, it is unclear whether group lattices can play a role as a possible space-time structure. In the Conclusion, we discuss the relation to compactified models and the possible implications for low-energy physics. An Appendix presents the proof that the representations obtained in Sect. 3 are irreducible and complete.

We follow the notation in Refs. [3, 5, 6]. A good introduction to the elementary number theory which we use is Ref. [10]. For integers a and b, $a|b$ indicates that a divides b, $\gcd(a, b)$ is the greatest common division of a and b, i. e., the biggest positive integer, c, such that $c|a$ and $c|b$, and $\mathrm{lcm}(a, b)$ is the least common multiple of a and b, that is, the smallest integer, c, such that $a|c$ and $b|c$. We work in Euclidean space. Minkowski space can be obtained, in principle, by Wick rotation: For scalar field theories, one changes the sign of the kinetic energy terms involving time derivatives and one includes a factor of i in the action.

Other notation is as follows. A subscript in parenthesis on a P or Q matrix indicates the size of the representation and signifies that commuting $P_{(N)}$ past $Q_{(N)}$ produces the phase $\exp\left(\frac{2\pi i}{N}\right)$, as in Eq. (2.1). The integer ℓ labels the prime-factor sectors. When it appears as a superscript, we enclose it in parenthesis to avoid confusion with a power. The primes appearing in the prime factorization are denoted by p_ℓ. The number of distinct primes is L. The variables r, s, t, u with subscripts or superscripts denote integer powers. The symbol r without a subscript indicates an irreducible representation. The variables i, j, k, l take on the values $1, 2, 3,$ and 4 and denote space-time directions. In the combination πi, i represents $\sqrt{-1}$ and not a space-time direction.

2 Prime Factorization

In this section, we introduce an idea which allows one to simplify the construction of the irreducible representations of the group associated with the d-dimensional twisted lattice. The point is to factorize the problem into a series of smaller parts.

Fundamental to the construction of twisted-group-lattice representations are the $N \times N$ matrices $P_{(N)}$ and $Q_{(N)}$. They satisfy the commutation relations

$$P_{(N)} Q_{(N)} = Q_{(N)} P_{(N)} \exp\left(\frac{2\pi i}{N}\right) \quad . \tag{2.1}$$

Explicit matrix forms for $P_{(N)}$ and $Q_{(N)}$ are [11]

$$P_{(N)} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad ,$$

$$Q_{(N)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & e^{2\pi i/N} & 0 & 0 & \dots & 0 \\ 0 & 0 & e^{4\pi i/N} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e^{2(N-1)\pi i/N} \end{pmatrix} \quad . \tag{2.2}$$

Suppose

$$N = N_1 N_2 \quad , \tag{2.3}$$

where N_1 and N_2 are relatively prime. It is an elementary result [10] of number theory that there then exist integers a_1 and a_2 such that

$$a_1 N_1 + a_2 N_2 = 1 \quad . \tag{2.4}$$

One necessarily has

$$\gcd(a_1, N_2) = \gcd(a_2, N_1) = 1 \quad . \tag{2.5}$$

An algorithm [10] exists for obtaining a_1 and a_2, given N_1 and N_2. When N_1 and N_2 are not large, it is usually easy to find a_1 and a_2 by inspection. From Eqs. (2.3) and (2.4), it follows that

$$\frac{1}{N} = \frac{a_2}{N_1} + \frac{a_1}{N_2} \quad . \tag{2.6}$$

Using this result and Eq. (2.1), it is easily shown that $P_{(N)}$ and $Q_{(N)}$ can be represented in tensor product form as

$$\begin{aligned} &Q_{(N_1)} \times Q_{(N_2)} \quad , \\ &P^{a_2}_{(N_1)} \times P^{a_1}_{(N_2)} \quad . \end{aligned} \tag{2.7}$$

Continuing the process, one factors N over the primes via

$$N = \prod_{\ell=1}^{L} p_\ell^{s_\ell} \quad , \tag{2.8}$$

where $p_1, p_2, \ldots, p_L$ are the L distinct primes appearing in N. The power of p_ℓ in N is s_ℓ. Again, by repeating the construction in the previous paragraph, there exist b_ℓ, $\ell = 1, 2, \ldots, L$, satisfying

$$\gcd\left(b_\ell, p_\ell^{s_\ell}\right) = 1 \quad , \tag{2.9}$$

and such that

$$\sum_{\ell=1}^{L} \frac{b_\ell}{p_\ell^{s_\ell}} = \frac{1}{N} \quad . \tag{2.10}$$

Then, $P_{(N)}$ and $Q_{(N)}$ can be represented as a tensor product of terms over the prime factors via

$$\begin{aligned} &Q_{\left(p_1^{s_1}\right)} \times Q_{\left(p_2^{s_2}\right)} \times \ldots \times Q_{\left(p_L^{s_L}\right)} \quad , \\ &P^{b_1}_{\left(p_1^{s_1}\right)} \times P^{b_2}_{\left(p_2^{s_2}\right)} \times \ldots \times P^{b_L}_{\left(p_L^{s_L}\right)} \quad . \end{aligned} \tag{2.11}$$

As side remark, we note that, since the representation of the algebra in Eq. (2.1) is unique, the matrices in Eqs. (2.2) and (2.11) are necessarily equivalent via a conjugation.

The advantage of using prime factorization is that the problem of constructing the irreducible representations for d-dimensional twisted group lattice also factorizes. In other words, it suffices to find irreducible representations for a single prime-factor sector.

3 Construction of the Irreducible Representations

The goal of this section is to find the complete set of irreducible representations for the group associated with the 4-d twisted lattice. With this knowledge, the problem of free propagation of particles on the lattice can be solved and a perturbative expansion of an interacting theory can be performed.

The parameters we use to characterize a representation are internal momenta k_{ij} and space-time momenta k_i. There is an irreducible representation r for each value of the "momenta" $k_{ij} = 0, 1, \ldots, N_{ij} - 1$ associated with the z_{ij} group elements. Here, the indices ij take on the values $ij = 12, 13, 14, 23, 24,$ and 34. When curly brackets $\{\ \}$ appear around a variable with ij indices, it means that ij range over these six values. Let d_r denote the dimension of the representation r. We choose the representation so that the matrices associated with z_{ij} are diagonal and proportional to the identity matrix $I_{(d_r)}$, i. e.,

$$z_{ij} \to I_{(d_r)} \exp\left(2\pi i \frac{k_{ij}}{N_{ij}}\right) \quad . \tag{3.1}$$

This is possible because the z_{ij} mutually commute. Note that the matrix representation of z_{ij} in Eq. (3.1) raised to the power N_{ij} yields the identity matrix, as it should, since $z_{ij}^{N_{ij}} = 1$.

Representations are also determined by the "momenta" $k_i = 0, 1, \ldots, L_i - 1$ associated with the elements x_i, where $i = 1, 2, 3$ or 4. However, some sets of $\{k_1, k_2, k_3, k_4\}$ lead to equivalent representations. Such representations are related by an element of a group E. Below we specify E in detail. The matrix representations of x_j are of the form [6]

$$x_j \to \Gamma_j \exp\left(2\pi i \frac{k_j}{L_j}\right) \quad , \tag{3.2}$$

where the Γ_i are $d_r \times d_r$ matrices satisfying

$$\Gamma_j \Gamma_i \ = \ \Gamma_i \Gamma_j \exp\left(2\pi i \frac{k_{ij}}{N_{ij}}\right) \ , \qquad \text{for } 1 \le i < j \le 4 \quad , \tag{3.3}$$

and

$$\Gamma_i^{L_i} = 1 \ , \qquad \text{for } i = 1, 2, 3, \text{ or } 4 \quad . \tag{3.4}$$

For the rest of this section, we assume that a specific set of $\{k_{ij}\}$ is given. Write

$$\frac{k_{ij}}{N_{ij}} = \frac{k'_{ij}}{N'_{ij}} \quad , \tag{3.5}$$

where the prime indicates that the fraction has been reduced, i. e., common factors have been cancelled so that

$$\gcd\left(k'_{ij}, N'_{ij}\right) = 1 \quad . \tag{3.6}$$

When $k_{ij} = 0$, we set $k'_{ij} = 0$ and $N'_{ij} = 1$. Note that it is only the combination $k_{ij}/N_{ij} = k'_{ij}/N'_{ij}$ which enters Eqs. (3.1) and (3.3) so that the primed variables are the quantities of interest.

Define

$$N' \equiv \mathrm{lcm}\left(\{N'_{ij}\}\right) = \mathrm{lcm}\left(N'_{12}, N'_{13}, N'_{14}, N'_{23}, N'_{24}, N'_{34}\right) \quad . \tag{3.7}$$

Perform a prime factorization of N'

$$N' = \prod_{\ell=1}^{L} p_\ell^{s_\ell} \ , \qquad \text{where } s_\ell \geq 1 \quad , \tag{3.8}$$

and of the N'_{ij}

$$N'_{ij} = \prod_{\ell=1}^{L} p_\ell^{t_{ij}^{(\ell)}} \ , \qquad \text{where } t_{ij}^{(\ell)} \geq 0 \quad . \tag{3.9}$$

The integer power $t_{ij}^{(\ell)}$ takes on the value zero when the prime p_ℓ does not appear in N'_{ij}. The exponent s_ℓ is the maximum value of $t_{ij}^{(\ell)}$ as ij ranges over $1 \leq i < j \leq 4$:

$$s_\ell = \max_{\{ij\}} \left\{ t_{ij}^{(\ell)} \right\} \quad . \tag{3.10}$$

Next define $b_{ij}^{(\ell)}$ by

$$\sum_{\ell=1}^{L} \frac{b_{ij}^{(\ell)}}{p_\ell^{s_\ell}} = \frac{k'_{ij}}{N'_{ij}} \quad . \tag{3.11}$$

Note that $b_{ij}^{(\ell)}$ may divide $p_\ell^{s_\ell}$ so that $\gcd\left(b_{ij}^{(\ell)}, p_\ell^{s_\ell}\right)$ is not necessarily 1. However, if $t_{ij}^{(\ell)} = s_\ell$ for specific ij then $\gcd\left(b_{ij}^{(\ell)}, p_\ell^{s_\ell}\right) = 1$. Thus, Eq. (3.10) implies that at least one $b_{ij}^{(\ell)}$ satisfies $\gcd\left(b_{ij}^{(\ell)}, p_\ell^{s_\ell}\right) = 1$.

The strategy is to solve the problem for a prime-factor sector ℓ. Write Γ_i as a tensor product via

$$\Gamma_i \ = \ \Gamma_i^{(1)} \times \Gamma_i^{(2)} \times \ldots \times \Gamma_i^{(L)} \quad . \tag{3.12}$$

If

$$\Gamma_j^{(\ell)} \Gamma_i^{(\ell)} \ = \ \Gamma_i^{(\ell)} \Gamma_j^{(\ell)} \exp\left(2\pi i \frac{b_{ij}^{(\ell)}}{p_\ell^{s_\ell}}\right) \quad , \tag{3.13}$$

and

$$\left(\Gamma_i^{(\ell)}\right)^{L_i} = 1 \quad , \tag{3.14}$$

then Eqs. (3.3) and (3.4) are satisfied due to Eqs. (3.5) and (3.11) – (3.14). Define

$$p_\ell^{r_\ell} \;\equiv\; \gcd\left(b_{12}^{(\ell)} b_{34}^{(\ell)} - b_{13}^{(\ell)} b_{24}^{(\ell)} + b_{14}^{(\ell)} b_{23}^{(\ell)} \;,\; p_\ell^{s_\ell}\right) \quad , \tag{3.15}$$

and let

$$u_\ell \equiv s_\ell - r_\ell \;, \qquad d_r^{(\ell)} \equiv p_\ell^{2s_\ell - r_\ell} = p_\ell^{s_\ell} p_\ell^{u_\ell} \quad . \tag{3.16}$$

The dimension of the matrix $\Gamma_i^{(\ell)}$ is

$$\dim \Gamma_i^{(\ell)} = d_r^{(\ell)} \quad . \tag{3.17}$$

The quantity $b_{12}^{(\ell)} b_{34}^{(\ell)} - b_{13}^{(\ell)} b_{24}^{(\ell)} + b_{14}^{(\ell)} b_{23}^{(\ell)}$ is related to the "obstruction" of the ℓ-sector matrices to be of minimum size. If

$$b_{12}^{(\ell)} b_{34}^{(\ell)} - b_{13}^{(\ell)} b_{24}^{(\ell)} + b_{14}^{(\ell)} b_{23}^{(\ell)} \;=\; 0 \pmod{p_\ell^{s_\ell}} \quad ,$$

then $u_\ell = 0$ and the minimum dimension $p_\ell^{s_\ell}$ occurs. As will be evident below, the matrices Γ_i are tensor products of two matrices of size $N'_\ell \times N'_\ell$ and $M_\ell \times M_\ell$, where

$$N'_\ell \equiv p_\ell^{s_\ell} \quad , \qquad M_\ell \equiv p_\ell^{u_\ell} \quad . \tag{3.18}$$

The total dimension is the product of the dimensions for each prime sector:

$$d_r = \prod_{\ell=1}^{L} d_r^{(\ell)} = \prod_{\ell=1}^{L} p_\ell^{s_\ell} p_\ell^{u_\ell} \quad . \tag{3.19}$$

The construction of the representation depends on which $b_{ij}^{(\ell)}$ has $\gcd\left(b_{ij}^{(\ell)}, p_\ell^{s_\ell}\right) = 1$. There are six cases

(1) $\gcd\left(b_{12}^{(\ell)}, p_\ell\right) = 1$,

(2) $\gcd\left(b_{13}^{(\ell)}, p_\ell\right) = 1$, but $\gcd\left(b_{12}^{(\ell)}, p_\ell\right) > 1$,

(3) $\gcd\left(b_{14}^{(\ell)}, p_\ell\right) = 1$, but $\gcd\left(b_{12}^{(\ell)}, p_\ell\right) > 1$ and $\gcd\left(b_{13}^{(\ell)}, p_\ell\right) > 1$,

(4) $\gcd\left(b_{23}^{(\ell)}, p_\ell\right) = 1$, but $\gcd\left(b_{1j}^{(\ell)}, p_\ell\right) > 1$, for $j = 2, 3$, and 4,

(5) $\gcd\left(b_{24}^{(\ell)}, p_\ell\right) = 1$, but $\gcd\left(b_{1j}^{(\ell)}, p_\ell\right) > 1$, for $j = 2, 3, 4$, and $\gcd\left(b_{23}^{(\ell)}, p_\ell\right) > 1$,

(6) $\gcd\left(b_{34}^{(\ell)}, p_\ell\right) = 1$, but $\gcd\left(b_{ij}^{(\ell)}, p_\ell\right) > 1$ for all $i < j$ except $ij = 34$.

We treat case (1) in detail. The other five cases are similar and can be obtained from case (1) by permuting space-time indices. For case (1),

$$\gcd\left(b_{12}^{(\ell)}, p_\ell^{s_\ell}\right) = 1 \quad . \tag{3.20}$$

Let

$$\begin{aligned} \Gamma_1^{(\ell)} &= Q_{\left(N'_\ell\right)} \times I_{(M_\ell)} \quad , \\ \Gamma_2^{(\ell)} &= P_{\left(N'_\ell\right)}^{b_{12}^{(\ell)}} \times I_{(M_\ell)} \quad . \end{aligned} \tag{3.21}$$

Equation (3.20) implies that there exist integers $a_3^{(\ell)}$ and $a_4^{(\ell)}$ such that

$$\begin{aligned} -b_{12}^{(\ell)} a_3^{(\ell)} &= b_{23}^{(\ell)} \pmod{p_\ell^{s_\ell}} \quad , \\ -b_{12}^{(\ell)} a_4^{(\ell)} &= b_{24}^{(\ell)} \pmod{p_\ell^{s_\ell}} \quad . \end{aligned} \tag{3.22}$$

Equations (3.15), (3.16) and (3.22) imply

$$\frac{b_{34}^{(\ell)} + a_4^{(\ell)} b_{13}^{(\ell)} - a_3^{(\ell)} b_{14}^{(\ell)}}{p_\ell^{s_\ell}} = \frac{b_4^{\prime(\ell)}}{p_\ell^{s_\ell - r_\ell}} = \frac{b_4^{\prime(\ell)}}{p_\ell^{u_\ell}} \quad , \tag{3.23}$$

where

$$\gcd\left(b_4^{\prime(\ell)}, p_\ell^{u_\ell}\right) = 1 \quad . \tag{3.24}$$

The two remaining matrices turn out to be given by

$$\begin{aligned} \Gamma_3^{(\ell)} &= Q_{\left(N'_\ell\right)}^{a_3^{(\ell)}} P_{\left(N'_\ell\right)}^{b_{13}^{(\ell)}} \times Q_{(M_\ell)} \exp\left[-i\pi\left(L_3 - 1\right)\frac{a_3^{(\ell)} b_{13}^{(\ell)}}{N'_\ell}\right] \quad , \\ \Gamma_4^{(\ell)} &= Q_{\left(N'_\ell\right)}^{a_4^{(\ell)}} P_{\left(N'_\ell\right)}^{b_{14}^{(\ell)}} \times P_{(M_\ell)}^{b_4^{\prime(\ell)}} \exp\left[-i\pi\left(L_4 - 1\right)\frac{a_4^{(\ell)} b_{14}^{(\ell)}}{N'_\ell}\right] \quad , \end{aligned} \tag{3.25}$$

where the phase factors ensure that $\left(\Gamma_i^{(\ell)}\right)^{L_i} = 1$.

It is straightforward to verify that the $\Gamma_i^{(\ell)}$ in Eqs. (3.21) and (3.25) satisfy Eq. (3.13): For $i = 1$ and $j = 2, 3, 4$, one immediately obtains

$$\Gamma_j^{(\ell)} \Gamma_1^{(\ell)} = \Gamma_1^{(\ell)} \Gamma_j^{(\ell)} \exp\left(2\pi i \frac{b_{1j}^{(\ell)}}{p_\ell^{s_\ell}}\right) \quad . \tag{3.26}$$

For $i = 2$ and $j = 3, 4$,

$$\Gamma_j^{(\ell)} \Gamma_2^{(\ell)} = \Gamma_2^{(\ell)} \Gamma_j^{(\ell)} \exp\left(-2\pi i \frac{b_{12}^{(\ell)} a_j^{(\ell)}}{p_\ell^{s_\ell}}\right) = \Gamma_2^{(\ell)} \Gamma_j^{(\ell)} \exp\left(2\pi i \frac{b_{2j}^{(\ell)}}{p_\ell^{s_\ell}}\right) \tag{3.27}$$

follows from Eq. (3.22). Finally,

$$\Gamma_4^{(\ell)}\Gamma_3^{(\ell)} = \Gamma_3^{(\ell)}\Gamma_4^{(\ell)} \exp\left(2\pi i \left(\frac{a_3^{(\ell)} b_{14}^{(\ell)} - a_4^{(\ell)} b_{13}^{(\ell)}}{p_\ell^{s_\ell}} + \frac{b_4'^{(\ell)}}{p_\ell^{u_\ell}}\right)\right)$$

$$= \Gamma_3^{(\ell)}\Gamma_4^{(\ell)} \exp\left(2\pi i \frac{b_{34}^{(\ell)}}{p_\ell^{s_\ell}}\right) \quad , \tag{3.28}$$

since

$$\frac{b_{34}^{(\ell)}}{p_\ell^{s_\ell}} = \frac{b_4'^{(\ell)}}{p_\ell^{u_\ell}} + \frac{a_3^{(\ell)} b_{14}^{(\ell)} - a_4^{(\ell)} b_{13}^{(\ell)}}{p_\ell^{s_\ell}} \quad . \tag{3.29}$$

Equation (3.29) is a consequence of Eq. (3.23).

All six cases lead to matrices of the form

$$\Gamma_i^{(\ell)} = Q_{(N_\ell')}^{a_i^{(\ell)}} P_{(N_\ell')}^{b_i^{(\ell)}} \times Q_{(M_\ell)}^{a_i'^{(\ell)}} P_{(M_\ell)}^{b_i'^{(\ell)}} \exp\left[-i\pi\left(L_i - 1\right)\left(\frac{a_i^{(\ell)} b_i^{(\ell)}}{N_\ell'} + \frac{a_i'^{(\ell)} b_i'^{(\ell)}}{M_\ell}\right)\right] \quad . \tag{3.30}$$

For example, for case (1)

$$\begin{array}{llll} a_1^{(\ell)} = 1 \ , & b_1^{(\ell)} = 0 \ , & a_1'^{(\ell)} = 0 \ , & b_1'^{(\ell)} = 0 \ , \\ a_2^{(\ell)} = 0 \ , & b_2^{(\ell)} = b_{12}^{(\ell)} \ , & a_2'^{(\ell)} = 0 \ , & b_2'^{(\ell)} = 0 \ , \\ a_3^{(\ell)} = a_3^{(\ell)} \ , & b_3^{(\ell)} = b_{13}^{(\ell)} \ , & a_3'^{(\ell)} = 1 \ , & b_3'^{(\ell)} = 0 \ , \\ a_4^{(\ell)} = a_4^{(\ell)} \ , & b_4^{(\ell)} = b_{14}^{(\ell)} \ , & a_4'^{(\ell)} = 0 \ , & b_4'^{(\ell)} = b_4'^{(\ell)} \ . \end{array} \tag{3.31}$$

A fairly compact formula for the dimension d_r can be obtained. Abbreviating

$$n_{ij}' \equiv \frac{N'}{N_{ij}'} \quad , \tag{3.32}$$

one has

$$d_r = \frac{N'^2}{\gcd\left(n_{12}' k_{12}' n_{34}' k_{34}' - n_{13}' k_{13}' n_{24}' k_{24}' + n_{14}' k_{14}' n_{23}' k_{23}' \ , \ N'\right)} \quad . \tag{3.33}$$

As remarked above, some $\{k_1, k_2, k_3, k_4\}$ lead to equivalent representations. Such momentum sets are related by the action of elements of a group E. The group E is generated by $4L$ elements. The generators are

$$E_{P\times I}^{(\ell)} : \text{ conjugation by } P_{(N_\ell')} \times I_{(M_\ell)} \text{ on the } \ell\text{th factor} \quad ,$$

$$E_{Q^{-1}\times I}^{(\ell)} : \text{ conjugation by } Q_{(N_\ell')}^{-1} \times I_{(M_\ell)} \text{ on the } \ell\text{th factor} \quad ,$$

$$E^{(\ell)}_{I\times P}: \text{ conjugation by } I_{\left(N'_\ell\right)} \times P_{(M_\ell)} \text{ on the } \ell\text{th factor} \quad ,$$

$$E^{(\ell)}_{I\times Q^{-1}}: \text{ conjugation by } I_{\left(N'_\ell\right)} \times Q^{-1}_{(M_\ell)} \text{ on the } \ell\text{th factor} \quad . \tag{3.34}$$

It is not difficult to see that the effect of these generators is to shift momenta by the following.

$$\begin{aligned}
&\text{For } E^{(\ell)}_{P\times I}: \qquad k_i \ \rightarrow \ k_i + \frac{a_i^{(\ell)} L_i}{N'_\ell} \quad (\text{mod } L_i) \quad , \\
&\text{for } E^{(\ell)}_{Q^{-1}\times I}: \qquad k_i \ \rightarrow \ k_i + \frac{b_i^{(\ell)} L_i}{N'_\ell} \quad (\text{mod } L_i) \quad , \\
&\text{for } E^{(\ell)}_{I\times P}: \qquad k_i \ \rightarrow \ k_i + \frac{a_i^{\prime(\ell)} L_i}{M_\ell} \quad (\text{mod } L_i) \quad , \\
&\text{for } E^{(\ell)}_{I\times Q^{-1}}: \qquad k_i \ \rightarrow \ k_i + \frac{b_i^{\prime(\ell)} L_i}{M_\ell} \quad (\text{mod } L_i) \quad .
\end{aligned} \tag{3.35}$$

For fixed ℓ, $p_\ell^{2s_\ell} p_\ell^{2u_\ell} = \left(d_r^{(\ell)}\right)^2$ identifications are made. The identifications are independent for different values of ℓ.

Hence, the irreducible representations are characterized by the six integer values of k_{ij}, which range from 0 to $N_{ij}-1$, and the values of the k_i modulo E. For fixed $\{k_{ij}\}$, the dimensions of the representations are the same, and the total number of identifications of the $\{k_1,k_2,k_3,k_4\}$ under E is equal to this dimension squared since

$$\prod_{\ell=1}^{L} p_\ell^{2s_\ell} p_\ell^{2u_\ell} = d_r^2 \quad . \tag{3.36}$$

Since each k_i ranges from 0 to $L_i - 1$, there are

$$\frac{L_1 L_2 L_3 L_4}{d_r^2} \tag{3.37}$$

independent values of $\{k_1,k_2,k_3,k_4\}$ for specific $\{k_{ij}\}$ (p_ℓ, s_ℓ, u_ℓ and d_r depend only on the k_{ij}). It follows that

$$\begin{aligned}
\sum_{(r)} d_r^2 \ &= \ \sum_{k_{12}=0}^{N_{12}-1} \cdots \sum_{k_{34}=0}^{N_{34}-1} \ \sum_{\{k_1,k_2,k_3,k_4\} \ (\text{mod } E)} d_r^2 \\
&= \ \sum_{k_{12}=0}^{N_{12}-1} \cdots \sum_{k_{34}=0}^{N_{34}-1} \sum_{k_1=0}^{L_1-1} \cdots \sum_{k_4=0}^{L_4-1} \frac{L_1 L_2 L_3 L_4}{d_r^2} d_r^2 \\
&= \ N_{12} N_{13} N_{14} N_{23} N_{24} N_{34} L_1 L_2 L_3 L_4 \ = \ o\left(G\right) \quad ,
\end{aligned} \tag{3.38}$$

where $o\left(G\right)$ is the order of G, i. e., the number of elements in G.

4 Field Theory Aspects

In this section, we consider field theories on group lattices. We first treat the propagation of a free particle on a four-dimensional twisted group lattice. The problem can be solved exactly using the results of Sect. 3. Secondly, we show how gauge theories can be defined on generic group lattices. The case of the two-dimensional twisted group lattice is solved exactly. Next, we consider anticommuting fields on the $\{N_{ij} = 2\}$ d-dimensional twisted group lattice. It was noted in Ref. [6] that Dirac-gamma-matrix structure naturally appears in the $\{k_{ij} = 1\}$ sector. We show that Kogut-Susskind staggered fermions, and not some other type of lattice fermions, arise. Finally, we analyze the space-time discrete symmetries of scalar field theories on a d-dimensional twisted group lattice.

This section provides the foundation for analyzing potentially interesting field theories on four-dimensional twisted group lattices. Standard-model-like particle theories can be treated. A gauge theory such as $SU(3) \times SU(2) \times U(1)$ can be defined on a four-dimensional twisted group lattice. Matter fields can also be introduced and coupled to gauge fields. A perturbative treatment of the system can be performed. The first step in a perturbative expansion is the solution of the free system.

4.1 The Free Theory Solution

It is straightforward to put field theories on a group lattice. Local field theories involve the product of fields at nearest neighbor sites and at the same site. For this reason, it is convenient to append e to $N\!N$, and define this set to be $N\!N_e$:

$$N\!N_e \equiv N\!N \cup \{e\} \quad . \tag{4.1}$$

Consider a single charged scalar field, ϕ. A free action S_0 for ϕ is

$$S_0 = \sum_{g \in G} \sum_{h \in N\!N_e} \lambda_h \phi^* (hg) \, \phi (g) \quad , \tag{4.2}$$

where λ_h are parameters. On regular lattices, one usually chooses $\lambda_e = 1$ and takes λ_h for $h \in N\!N$ to be negative, in which case the λ_h are called hopping parameters. The field theory in Eq. (4.2) leads to the propagation of a charged particle through the group lattice. The hopping parameter $-\lambda_h$ controls how easily or difficult it is for the charged particle to propagate over the bond $[hg, g]$.

The path integral Z for this system is given by

$$Z = \int \mathcal{D}\phi\mathcal{D}\phi^* \exp\left(-S_0\right) \quad , \tag{4.3}$$

where the measure is

$$\mathcal{D}\phi\mathcal{D}\phi^* = \prod_{g\in G} \frac{d\phi\left(g\right) d\phi^*\left(g\right)}{2\pi} \quad . \tag{4.4}$$

Free propagation on a group lattice is exactly solvable if all the irreducible matrix representations of the group G are known [3]. The solution method is based on the group analog of the Fourier transform. The result for the action in Eq. (4.2) is

$$Z = \prod_r \left[\det\left(\sum_{h\in NN_e} \lambda_h D^{(r)}\left(h\right)\right)\right]^{-d_r} \quad , \tag{4.5}$$

where $D^{(r)}\left(h\right)$ is the matrix for h in the irreducible representation r and d_r is the dimension of the representation. The product in Eq. (4.5) is over all the irreducible representations r of G. As long as the dimensions of the representations are not too big, the determinants which enter in Eq. (4.5) are of moderate size and can be straightforwardly computed. For a twisted group lattice, the matrix in Eq. (4.5) is

$$\sum_{h\in NN_e} \lambda_h D^{(r)}\left(h\right) \;=\; \lambda_e I_{d_r} +$$

$$\sum_{j=1}^{d}\left(\lambda_h \Gamma_j^{(r)} \exp\left(\frac{2\pi i k_j^{(r)}}{L_j}\right) + \lambda_{h^{-1}}\left[\Gamma_j^{(r)}\right]^{-1} \exp\left(\frac{-2\pi i k_j^{(r)}}{L_j}\right)\right) \quad , \tag{4.6}$$

where the $\Gamma_j^{(r)}$ are the matrices satisfying Eq. (3.3) in the irreducible representation r and the $k_j^{(r)}$ are the corresponding spatial momenta. For the four-dimensional twisted group lattice, the $\Gamma_j^{(r)}$ are given in Sect. 3, and the product over representations in Eq. (4.5) is

$$\prod_r = \prod_{\{k_{ij}=0\}}^{\{N_{ij}-1\}} \prod_{\left\{k_i^{(r)}\right\} \pmod{E}} \quad . \tag{4.7}$$

Interactions are introduced by considering products involving more than two fields. For example, a "ϕ-fourth" theory is defined by adding S_{int}, given by

$$S_{int} = \lambda \sum_{g\in G} \phi^*\left(g\right)\phi\left(g\right)\phi^*\left(g\right)\phi\left(g\right) \quad , \tag{4.8}$$

to S_0, where λ is a coupling constant. It is straightforward to obtain propagators for this theory [3]. Hence, perturbation theory via Feynman graphs can be carried out.

Fermionic field theories are obtained by making ϕ anticommuting. The above equations still hold with the exception of Eq. (4.5) where the power of the matrix is of the opposite sign, i. e., one replaces $-d_r$ by d_r in Eq. (4.5). One could also consider a real scalar field theory. In this case, $\phi^*(hg) \to \phi(hg)$ in Eq. (4.2), and the measure in Eq. (4.4) becomes $\mathcal{D}\phi\mathcal{D}\phi^* \to \prod_{g\in G} \frac{d\phi(g)}{\sqrt{2\pi}}$. The solution is Eq. (4.5) with $d_r \to d_r/2$, and the matrix in Eq. (4.6) is the same except $\lambda_h \to (\lambda_h + \lambda_{h^{-1}})/2$ and $\lambda_{h^{-1}} \to (\lambda_h + \lambda_{h^{-1}})/2$.

4.2 Lattice Gauge Theories on Group Lattices

It is straightforward to put gauge theories on a group lattice, as soon as the plaquettes in the lattice are specified. One selects a gauge group $\mathcal{G}$. Frequent choices for $\mathcal{G}$ are $SU(M)$, $U(M)$, and $SO(M)$. One can even use a discrete gauge group, although it is more difficult or impossible to take a continuum limit in this case for certain systems for which a thermodynamic limit is possible.[3]

To put a gauge theory on a lattice, one uses the standard Wilson method. For each bond $[hg, g]$ in the lattice, one assigns a "link variable" $U_{hg,g}$ in a matrix representation $R_{\mathcal{G}}$ of the gauge group, i. e.,

$$U_{hg,g} \in R_{\mathcal{G}} \ , \qquad \text{for all } g \in G \text{ and } h \in N\!\!N \quad . \tag{4.9}$$

Often $R_{\mathcal{G}}$ is the fundamental representation. A link variable, which is oriented in the opposite direction, is the inverse matrix, i. e.,

$$U_{g,hg} = U^{-1}_{hg,g} \quad . \tag{4.10}$$

Usually a unitary representation is used so that $U^{-1}_{hg,g} = U^{\dagger}_{hg,g}$. Equation (4.10) says that a link variable for a bond using a shift h^{-1} is the inverse of the link variable using a shift h but at the site $h^{-1}g$:

$$U_{h^{-1}g,g} = U^{-1}_{hg',g'} \ , \qquad \text{where } g' = h^{-1}g \quad . \tag{4.11}$$

For this reason it is not necessary to assign link variables for the bonds associated with inverse elements in $N\!\!N$. If h is its own inverse, $h = h^{-1}$, then the corresponding link variable must be its own inverse. Usually there are only discrete choices for this case and a continuum limit may not be possible. Such elements h do not occur for the d-dimensional twisted group lattices.

[3]Thermodynamic limits can be considered for twisted group lattices by letting $L_i \to \infty$.

A plaquette p is a series of bonds that form a closed loop in the lattice. We denote the set of all plaquettes by $\mathcal{P}$. The choice of $\mathcal{P}$ is at one's disposal. If $p \in \mathcal{P}$, then p is determined by the starting point g and by a series of elements $h_1, h_2, \ldots, h_m$ in NN, where $h_m h_{m-1} \cdots h_1 = e$. The corresponding closed path begins at g, goes to $h_1 g$, then to $h_2 h_1 g$, …, then to $h_{m-1} h_{m-2} \cdots h_1 g$, and back to g. Plaquettes which transverse the same path but start at different points are considered equivalent. A plaquette variable U_p is computed by multiplying the link variables around the path. More precisely,

$$U_p \;=\; U_{g,h_{m-1}h_{m-2}\ldots h_1 g} U_{h_{m-1}h_{m-2}\ldots h_1 g, h_{m-2}\ldots h_1 g} \cdots U_{h_2 h_1 g, h_1 g} U_{h_1 g, g} \quad . \tag{4.12}$$

The gauge theory action $S_{\mathcal{G}}$ is then given by

$$S_{\mathcal{G}} = \beta \sum_{p \in \mathcal{P}} \left(Tr\left[U_p\right] + Tr\left[U_p^{-1}\right] \right) \quad , \tag{4.13}$$

where β is a coupling constant, and Tr is the matrix trace, i. e., the character of U_p in the representation $R_{\mathcal{G}}$. The partition function is

$$Z_{\mathcal{G}}\left(\beta\right) = \left(\prod_{b \in \mathcal{B}} \int dU_b \right) \exp(-S_{\mathcal{G}}) \quad , \tag{4.14}$$

where the measure involves the product over all bonds in the lattice. In Eq. (4.14), dU_b is the left and right group-invariant Haar measure, normalized so that $\int dU_b = 1$. The theory governed by Eq. (4.13) has the gauge invariance

$$U_{hg,g} \rightarrow V_{hg} U_{hg,g} V_g^{-1} \quad , \tag{4.15}$$

where $V_g \in R_{\mathcal{G}}$.

Matter fields which interact with gauge fields can be introduced. Let ϕ be a field which transform under the gauge group via $\phi(g) \rightarrow R_{(\phi)}\left[V_g\right] \phi(g)$, for some unitary representation $R_{(\phi)}$ of the gauge group. The action in Eq. (4.2) is then modified to

$$S = \sum_{g \in G} \sum_{h \in NN_e} \lambda_h \phi^*\left(hg\right) R_{(\phi)}\left[U_{hg,g}\right] \phi\left(g\right) \quad . \tag{4.16}$$

It is gauge invariant as long as $U_{hg,g}$ is transformed as in Eq. (4.15). We define $U_{g,g} = 1$ so that Eq. (4.16) makes sense when $h = e$.

Applying the above construction to the d-dimensional twisted group lattice, one obtains the following. There exist link variables $U_{x_i g,g}$ for all i and all g. Eq. (4.11) for $U_{x_i^{-1} g,g}$ gives $U_{x_i^{-1} g,g} = U^{-1}_{g,x_i^{-1} g}$. A natural choice for the plaquette set $\mathcal{P}$ exists. On

regular hypercubic lattices, for which all $N_{ij} = 1$, plaquettes are obtained by using the elements x_i, x_j, x_i^{-1} and x_j^{-1} for any $i < j$. In other words, the four corners of a plaquette in the i-j plane are g, $x_i g$, $x_j x_i g$ and $x_i^{-1} x_j x_i g$. On a twisted group lattice, however, such a path is not closed since, when one moves from the last site $x_i^{-1} x_j x_i g$ using x_j^{-1}, one does not return to the original site g. Instead, one arrives at $x_j^{-1} x_i^{-1} x_j x_i g = z_{ij} g$. On a twisted group lattice, one must go around the ij square N_{ij} times to return to the original site. We choose such paths for all $i < j$ to be the plaquettes. In other words, the corners of a plaquette are

$$p \leftrightarrow \Big\{ x_i^{-1} x_j x_i z_{ij}^{N_{ij}-1} g, x_j x_i z_{ij}^{N_{ij}-1} g, x_i z_{ij}^{N_{ij}-1} g, z_{ij}^{N_{ij}-1} g, \ldots, \\ x_i z_{ij} g, z_{ij} g, x_i^{-1} x_j x_i g, x_j x_i g, x_i g, g \Big\} \quad . \tag{4.17}$$

The set $\mathcal{P}$ for the d-dimensional twisted group lattice is obtained from Eq. (4.17) by using all g and all $i < j$. The action and partition function are then given as in Eqs. (4.13) and (4.14).

Non-trivial twisted group lattices, for which some $N_{ij} > 1$, possess link gauge invariances. The situation is similar to the adjoint action $S_{\mathcal{G}} = \beta_A \sum_{p\in\mathcal{P}} \left(Tr\left[U_p\right] Tr\left[U_p^\dagger\right] \right)$ of an $SU(M)$ lattice gauge theory on a regular lattice. If $U_p \to V U_p$ for any V in the center Z_M of $SU(M)$ then $S_{\mathcal{G}}$ is left unchanged. This is a local symmetry since the transformation can be performed on any link. Hence, the effective gauge group is $SU(M)/Z_M$. Using the adjoint action in lieu of the Wilson action, which is based on the fundamental representation, has a dramatic effect on the physics. A non-overlapping Wilson loop in the fundamental representation vanishes. This result follows from the above local link symmetry.

To determine the additional local invariances of the gauge theory on a d-dimensional twisted group lattice, it is useful to define some concepts. The center $\mathcal{Z}_G$ of the group G governing the group lattice in Eq. (1.1) is generated by the z_{ij} elements, i. e.,

$$\mathcal{Z}_G = \left\{ \prod_{i<j} z_{ij}^{n_{ij}} \text{ , such that } n_{ij} = 0, 1, \ldots, N_{ij} - 1 \right\} \quad . \tag{4.18}$$

Let us say that g is *above or below* g' if $g = zg'$ for some $z \in \mathcal{Z}_G$. A bond $[x_i g, g]$ is *above or below* the bond $[x_i g', g']$ if g is above or below g'. Let g be a fixed point on the group lattice and fix i. Consider the transformation

$$U_{x_i zg, zg} \to V_z U_{x_i zg, zg} \quad , \tag{4.19}$$

for V_z in the center $\mathcal{Z}_{\mathcal{G}}$ of the gauge group representation $R_{\mathcal{G}}$. Equation (4.19) is a local transformation for all link variables on the bonds above or below the bond

$[x_i g, g]$. Let $\mathcal{I}$ denote the identity matrix in the representation $R_{\mathcal{G}}$. If for every $z \in \mathcal{Z}_G$, one has

$$\prod_{n=0}^{N_{ij}-1} V_{z_{ij}^n z} = \mathcal{I} \ , \qquad \text{for every } j > i \quad (i \text{ fixed}) \quad ,$$

$$\prod_{n=0}^{N_{ji}-1} V_{z_{ji}^n z} = \mathcal{I} \ , \qquad \text{for every } j < i \quad (i \text{ fixed}) \quad , \tag{4.20}$$

then the plaquette variables TrU_p are left unchanged, so that the transformation in Eq. (4.19) is a local symmetry. For $d = 2$, the condition in Eq. (4.20) simplifies to

$$\prod_{n=0}^{N_{12}-1} V_{z_{12}^n} = \mathcal{I} \quad . \tag{4.21}$$

It is easy to find solutions to Eq. (4.21). For example, $U_{x_1,e} \to VU_{x_1,e}$, $U_{z_{12}x_1,z_{12}} \to V^{-1}U_{z_{12}x_1,z_{12}}$, where $V \in \mathcal{Z}_{\mathcal{G}}$, produces a local link symmetry.

The behavior of gauge theories on four-dimensional twisted group lattices is an interesting topic which goes beyond the current work and may require the use of computer simulations. Important issues are the question of confinement, asymptotic freedom, and the continuum limit. Due to the local link invariance, the nature of confinement, if it exists, might be different in the sense that certain Wilson loops may have zero expectation value. The link gauge symmetry may be broken, if so desired, by introducing additional or different plaquettes in $\mathcal{P}$. It can also be broken by introducing matter fields which couple to the gauge fields as in Eq. (4.16).

Two-dimensional gauge theories on *regular* lattices are exactly solvable. The strong coupling expansion can be computed and gives results valid for all values of the coupling. On each plaquette, one performs the following character expansion

$$\exp\left[\beta Tr\left(U_p\right) + \beta Tr\left(U_p^{\dagger}\right)\right] = \mathcal{Z}_0\left(\beta\right) \sum_{(\rho)} d_\rho z_\rho\left(\beta\right) \chi_\rho\left(U_p\right) \quad , \tag{4.22}$$

where the sum is over all irreducible representations (ρ) of the gauge group $\mathcal{G}$. In Eq. (4.22), χ_ρ and d_ρ denote the character and dimension of ρ. The single plaquette partition function $\mathcal{Z}_0\left(\beta\right)$ is given by

$$\mathcal{Z}_0\left(\beta\right) = \int dU \exp\left[\beta Tr\left(U\right) + \beta Tr\left(U^{\dagger}\right)\right] \quad . \tag{4.23}$$

Explicit formulas for $\mathcal{Z}_0\left(\beta\right)$ and the character coefficients $z_\rho\left(\beta\right)$ can be found in many places. See, for example, Ref. [12]. After the expansion in Eq. (4.22) is performed,

integration over link variables can be explicitly done using the integral formulas

$$\int dU \chi_\rho (AU) \chi_\sigma \left(U^\dagger B\right) = \frac{\delta_{\rho\sigma}}{d_\rho} \chi_\rho (AB) \quad ,$$

$$\int dU \chi_\rho \left(AUBU^\dagger\right) = \frac{1}{d_\rho} \chi_\rho (A) \chi_\rho (B) \quad , \tag{4.24}$$

which follow from the orthogonality relations for integrals over irreducible representations. Only two characters appear in performing the strong coupling integrations because each bond is shared by only two plaquettes for a two-dimensional surface. The final result is

$$Z_{\mathcal{G}} (\beta) = [\mathcal{Z}_0 (\beta)]^V \sum_{(\rho)} d_\rho^{2-2g} [z_\rho (\beta)]^V \quad , \tag{4.25}$$

where $V = L_1 L_2$ is the volume of the system, i. e., the number of plaquettes, and g is the genus of the surface.

Gauge theories on *twisted* two-dimensional group lattices are also exactly solvable. With the above definition of $\mathcal{P}$, a two-dimensional twisted group lattice is a closed two-dimensional surface since each bond is shared by exactly two plaquettes. Gluing plaquettes together at such bonds produces the two-dimensional surface. Since Eq. (4.25) holds for an arbitrary two-dimensional lattice, one can compute the genus of the periodic twisted group lattice using lattice gauge theory. Orient the lattice so that the x and y axes point respectively in the horizontal and vertical directions. Consider the plaquettes along a horizontal strip of length N_{12}. Perform the integrations of the link variables on all interior vertical bonds. There are $N_{12} (N_{12} - 1)$ such bonds. These integrations yield a factor of $d_\rho^{-N_{12}(N_{12}-1)}$ via Eq. (4.24) and produce N_{12} separate Wilson-type loops. One can do this for all strips of size $1 \times N_{12}$. The result is N_{12} separate gauge theories on regular toroidal lattices with spatial volumes of V/N_{12}. The remaining integrations then proceed as for the regular lattice case. Note that there are also factors of $[z_\rho (\beta)]^V$ and $[\mathcal{Z}_0 (\beta)]^V$ from Eq. (4.22), where V is the space-time volume: $V = L_1 L_2$. Hence, one finds

$$Z_{N_{12}} (\beta) = [\mathcal{Z}_0 (\beta)]^V \sum_{(\rho)} d_\rho^{(1-N_{12})V} [z_\rho (\beta)]^V \quad . \tag{4.26}$$

One concludes that the genus g is

$$g = \frac{(N_{12} - 1) V}{2} + 1 \quad . \tag{4.27}$$

The genus is generally quite large and grows with the space-time volume. A two-dimensional twisted group lattice is indeed quite twisted. When $N_{12} = 1$, one recovers the ordinary lattice-gauge-theory result on the torus.

Equation (4.27) can be computed using Euler's theorem which says that $2(1-g) = v - e + f$, where v, e and f are respectively the number of vertices, edges and faces of the lattice. For the twisted group lattice, $v = N_{12}V$, $e = 2N_{12}V$ and $f = V$.

"Planar" Wilson loops can be evaluated. They either vanish due to the local link symmetry or have area law. Confinement is a property of gauge theories on two-dimensional twisted group lattices.

4.3 Automatic Dirac-Gamma-Matrix Structure

It was noted in Ref. [6] that the $\{k_{ij} = 1\}$ sector of the $\{N_{ij} = 2\}$ case leads to Dirac-gamma-matrix structure. If anti-commuting fields are permitted to propagate on this twisted group lattice, then Dirac fermions emerge. It is interesting to determine whether these fermions are naive, staggered, or of some other type.

To analyze the situation, it is convenient to represent the d-dimensional twisted group lattice as a $d(d+1)/2$-dimensional lattice in which there are screw dislocation defects at the centers of all plaquettes [5, 6]. One starts with the $d(d+1)/2$-dimensional periodic lattice

$$L_1 \times L_2 \times \ldots \times L_d \times N_{12} \times N_{13} \times \ldots \times N_{d-1d} \quad . \tag{4.28}$$

Let

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d, \mathbf{e}_{12}, \mathbf{e}_{13}, \ldots, \mathbf{e}_{d-1d} \quad , \tag{4.29}$$

be the standard orthonormal basis. The first d basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ correspond to the usual d directions. There are plaquettes in each of the $d(d-1)/2$ planes. One arbitrarily appends $d(d-1)/2$ directions associated with these planes. The remaining $d(d-1)/2$ vectors $\mathbf{e}_{12}, \mathbf{e}_{13}, \ldots, \mathbf{e}_{d-1d}$ are a basis for this auxiliary space. A point $\mathbf{y}$ in ordinary space corresponds to

$$\mathbf{y} = \sum_{i=1}^{d} y_i \mathbf{e}_i \quad . \tag{4.30}$$

A point in the full space is denoted by $(\mathbf{y}, \mathbf{w})$ where

$$\mathbf{w} = \sum_{1 \le i < j \le d} w_{ij} \mathbf{e}_{ij} \quad , \tag{4.31}$$

that is,

$$(\mathbf{y}, \mathbf{w}) = \sum_{i=1}^{d} y_i \mathbf{e}_i + \sum_{1 \le i < j \le d} w_{ij} \mathbf{e}_{ij} \quad . \tag{4.32}$$

Screw dislocations are introduced by modifying the direction of bonds in the i-direction, for $i = 1, \ldots, d-1$. Bonds exist between $(\mathbf{y}, \mathbf{w})$ and $\Big(\mathbf{y}{+}\mathbf{e}_i, \mathbf{w}{-}\sum\limits_{j=i+1}^{d} \mathbf{e}_{ij} y_j\Big)$. A movement along this bond is associated with x_i. A movement between $(\mathbf{y}, \mathbf{w})$ and $(\mathbf{y}, \mathbf{w}{+}\mathbf{e}_{ij})$ is associated with z_{ij}. It is easy to check that, for $j > i$, an x_i movement, followed by an x_j movement is an x_j movement, followed by an x_i movement, followed by a z_{ij} movement. Noting that we order "movements" from right to left, the last statement corresponds to a physical realization of the equation $x_j x_i = z_{ij} x_i x_j$ for $j > i$.

Let ψ be an anticommuting field and let $\bar{\psi}$ be its conjugate. Consider the following action

$$A \;=\; \sum_{\mathbf{y},\mathbf{w}} \sum_{i=1}^{d} \left(\lambda_{\mathbf{e}_i} \bar{\psi}\Big(\mathbf{y}{+}\mathbf{e}_i, \mathbf{w}{-} \sum_{j=i+1}^{d} \mathbf{e}_{ij} y_j\Big)\, \psi\left(\mathbf{y}, \mathbf{w}\right) \;+\right.$$

$$\left. \lambda_{-\mathbf{e}_i} \bar{\psi}\Big(\mathbf{y}{-}\mathbf{e}_i, \mathbf{w}{+} \sum_{j=i+1}^{d} \mathbf{e}_{ij} y_j\Big) \psi\left(\mathbf{y}, \mathbf{w}\right) \right) \;-\; \sum_{\mathbf{y},\mathbf{w}} \lambda_0 \bar{\psi}\left(\mathbf{y}, \mathbf{w}\right) \psi\left(\mathbf{y}, \mathbf{w}\right) \quad , \qquad (4.33)$$

where $\lambda_{\mathbf{e}_i}$ and $\lambda_{-\mathbf{e}_i}$ are hopping parameters for the shifts associated with x_i and x_i^{-1}. The parameter λ_0, which is λ_h for $h = e$, is a chemical potential or a mass. Note that the arguments in $\bar{\psi}$ correspond to the shifts created by x_i, x_i^{-1} or e.

Perform a Fourier transformation with respect to the $d(d-1)/2$ variables $\{w_{ij}\}$ via

$$\psi\left(\mathbf{y}, \mathbf{w}\right) \;=\; \left(\prod_{1 \le i < j \le d} \frac{1}{\sqrt{N_{ij}}} \sum_{k_{ij}=0}^{N_{ij}-1} \right) \exp\left[2\pi i \sum_{1 \le i < j \le d} \frac{k_{ij} w_{ij}}{N_{ij}} \right] \tilde{\psi}\left(\mathbf{y}, \{k_{ij}\}\right) \quad ,$$

$$\bar{\psi}\left(\mathbf{y}, \mathbf{w}\right) \;=\; \left(\prod_{1 \le i < j \le d} \frac{1}{\sqrt{N_{ij}}} \sum_{k_{ij}=0}^{N_{ij}-1} \right) \exp\left[-2\pi i \sum_{1 \le i < j \le d} \frac{k_{ij} w_{ij}}{N_{ij}} \right] \tilde{\bar{\psi}}\left(\mathbf{y}, \{k_{ij}\}\right) \quad , \quad (4.34)$$

where the transformed fields are denoted by a tilde. The action becomes

$$A \;=\; \sum_{\{k_{ij}=0\}}^{\{N_{ij}-1\}} \sum_{\mathbf{y}} \sum_{i=1}^{d} \Big(\lambda_{\mathbf{e}_i} \alpha_i \tilde{\bar{\psi}}\left(\mathbf{y}{+}\mathbf{e}_i, \{k_{ij}\}\right) \tilde{\psi}\left(\mathbf{y}, \{k_{ij}\}\right) +$$

$$\lambda_{-\mathbf{e}_i} \alpha_i^{-1} \tilde{\bar{\psi}}\left(\mathbf{y}{-}\mathbf{e}_i, \{k_{ij}\}\right) \tilde{\psi}\left(\mathbf{y}, \{k_{ij}\}\right) \Big) - \sum_{\mathbf{y},\{k_{ij}\}} \lambda_0 \tilde{\bar{\psi}}\left(\mathbf{y}, \{k_{ij}\}\right) \tilde{\psi}\left(\mathbf{y}, \{k_{ij}\}\right) \quad , \quad (4.35)$$

where

$$\alpha_i \equiv e^{2\pi i \sum\limits_{j=i+1}^{d} \frac{k_{ij} y_j}{N_{ij}}} \quad . \qquad (4.36)$$

The theory has factorized into sectors corresponding to the $\{k_{ij}\}$.

Consider the case $N_{ij} = 2$ and the subsector $k_{ij} = 1$ for all $1 \le i < j \le d$ and let $\lambda_{\mathbf{e}_i} = -\lambda_{-\mathbf{e}_i} = \frac{i}{2}$ for $i = 1, 2, \ldots, d$. The phase factors α_i in Eq. (4.26) become

$$\alpha_i \to (-1)^{\sum_{j=i+1}^{d} y_j} \; , \qquad \alpha_i^{-1} \to (-1)^{\sum_{j=i+1}^{d} y_j} \quad . \tag{4.37}$$

Let χ be the $\{k_{ij} = 1\}$ Fourier component of ψ:

$$\chi\left(\mathbf{y}\right) = \tilde{\psi}\left(\mathbf{y}, \{k_{ij} = 1\}\right) \; , \qquad \bar{\chi}\left(\mathbf{y}\right) = \tilde{\bar{\psi}}\left(\mathbf{y}, \{k_{ij} = 1\}\right) \quad . \tag{4.38}$$

The action $A_{\{k_{ij}=1\}}$ in this sector is

$$A_{\{k_{ij}=1\}} = \sum_{\mathbf{y}} \sum_{i=1}^{d} \frac{i}{2} \alpha_i \left(\bar{\chi}\left(\mathbf{y}{+}\mathbf{e}_i\right) \chi\left(\mathbf{y}\right) - \bar{\chi}\left(\mathbf{y}{-}\mathbf{e}_i\right) \chi\left(\mathbf{y}\right) \right) - \lambda_0 \sum_{\mathbf{y}} \bar{\chi}\left(\mathbf{y}\right) \chi\left(\mathbf{y}\right) \; , \tag{4.39}$$

where $\alpha_i = (-1)^{\sum_{j=i+1}^{d} y_j}$ is a sign factor appropriate for Kogut-Susskind staggered fermions [13]. Hence, the $\{k_{ij} = 1\}$ sector of the $\{N_{ij} = 2\}$ twisted group lattice contains staggered fermions.

4.4 Space-Time Discrete Symmetries

Regular hypercubic lattices have discrete space-time symmetries, such as translations by the lattice spacing and 90° rotations in a plane. The same is true for the twisted d-dimensional group lattices. Let g' be a fixed element. Field theories on group lattices possess the symmetry [3]

$$g \to g g' \quad \text{for all } g \quad . \tag{4.40}$$

For example, in the case of a scalar field, ϕ, the transformation $\phi(g) \to \phi(gg')$ leaves the action invariant because the transformation in Eq. (4.40) maintains the bond structure: if g_1 and g_2 are nearest neighbor sites, then $g_1 g'$ and $g_2 g'$ are also nearest neighbor sites. Note that one must multiply g' from the right since nearest neighbor sites are determined by multiplication from the left. A translation in the ith direction by m_i lattice spacings corresponds to Eq. (4.40) for $g' = x_i^{m_i}$, i. e., $g \to g x_i^{m_i}$. When an element is represented as in Eq. (1.1), m_i is added to the exponent n_i of x_i while leaving the exponents of the other x_j unchanged. Hence $g \to g x_i^{m_i}$ corresponds to a shift $n_j \to n_j + \delta_j^i m_i$. Of course, z_{ij} exponents may change, but this is a necessary

consequence of twisting. One also has translations in the "extra dimensions" via $g \to g z_{ij}^{m_{ij}}$, but these can be generated by translations in i and j directions using $x_j^{-1} x_i^{-1} x_j x_i$ repeatedly m_{ij} times. The translation group of the twisted lattice is different from the one of the regular hypercubic lattice because of twisting. It is precisely this difference which makes the twisted group lattice interesting.

Regular hypercubic lattices possess 90° rotational symmetries. Let R_{ij} denote a rotation of the ij-plane by 90°. Representing the regular lattice as an abelian group lattice, one defines R_{ij} by

$$\begin{aligned} R_{ij}\,[e] = e \ , \qquad & R_{ij}\,[x_i] = x_j \quad , \\ R_{ij}\,[x_j] = x_i^{-1} \ , \qquad & R_{ij}\,[x_k] = x_k \quad , \end{aligned} \tag{4.41}$$

where i, j, and k are distinct, and where we denote the action of R_{ij} on a point g by $R_{ij}\,[g]$. On a general element, R_{ij} is determined from Eq. (4.41) and the requirement that it be a homomorphism

$$R_{ij}\,[g_1 g_2] = R_{ij}\,[g_1]\, R_{ij}\,[g_2] \ , \qquad R_{ij}\left[g^{-1}\right] = \left(R_{ij}\,[g]\right)^{-1} \quad . \tag{4.42}$$

Discrete rotations satisfy the group relations

$$\begin{aligned} R_{ij} R_{ji} = E \ , \qquad & (R_{ij})^4 = E \quad , \\ R_{jk} R_{ki} R_{kj} = R_{ij} \ , \qquad R_{ki} R_{kj} R_{ik} = R_{ij} \ , \qquad & R_{ij} R_{kl} = R_{kl} R_{ij} \quad , \end{aligned} \tag{4.43}$$

where i, j, k and l are distinct, and E is the identity transformation. Equation (4.43) is the lattice analog of the continuum $O(d)$ Lie group relations.

The twisted d-dimensional group lattices also possess discrete rotational symmetries if the lengths L_i in all directions are equal: $L_i = L$, and the twistings are all equal: $N_{ij} = N$ for $i < j$. We define the action R_{ij} on group elements via Eqs. (4.41) and (4.42)[4] and by requiring R_{ij} to respect the group relations. This implies the additional actions

$$\begin{aligned} R_{ij}\,[z_{ij}] = z_{ij} \ , \qquad & R_{ij}\,[z_{kl}] = z_{kl} \quad , \\ R_{ij}\,[z_{ik}] = z_{jk} \ , \qquad & R_{ij}\,[z_{ki}] = z_{kj} \quad , \\ R_{ij}\,[z_{jk}] = z_{ik}^{-1} \ , \qquad & R_{ij}\,[z_{kj}] = z_{ki}^{-1} \quad , \end{aligned} \tag{4.44}$$

where i, j, k and l are distinct, and where, for notational convenience, we have defined $z_{ji} \equiv z_{ij}^{-1}$ for $j > i$. The group relations are all satisfied: Since $e = x_l^L$, one needs $e =$

[4] It is important to maintain the order in Eq. (4.42) since now the group is non-abelian.

$R_{ij}[e] \stackrel{?}{=} R_{ij}\left[x_i^L\right] = x_j^L \stackrel{\checkmark}{=} e$, $e \stackrel{?}{=} R_{ij}\left[x_j^L\right] = x_i^{-L} \stackrel{\checkmark}{=} e$, and $e \stackrel{?}{=} R_{ij}\left[x_k^L\right] = x_k^L \stackrel{\checkmark}{=} e$. These equations show why all L_i must be equal. Likewise, the relation $x_j^{-1}x_i^{-1}x_jx_i = z_{ij}$ is respected by rotations if Eq. (4.44) holds. As long as $N_{ij} = N$ for all $i < j$, $z_{ij}^{N_{ij}} = e$ is preserved by rotations.

Reflections can be defined in a manner analogous to rotations. The reflection of the ith coordinate corresponds to an operator R_i satisfying

$$R_i[x_i] = x_i^{-1}\ , \qquad R_i[x_j] = x_j \quad ,$$

$$R_i[z_{ij}] = z_{ij}^{-1}\ , \qquad R_i[z_{ji}] = z_{ji}^{-1}\ , \qquad R_i[z_{jk}] = z_{jk} \quad , \tag{4.45}$$

where i, j and k are distinct.

The discrete rotations and reflections are symmetries of scalar field theories on twisted group lattices because the bond structure is maintained. Of course, the couplings λ_h for $h \neq e$ in Eq. (4.2) must all be equal: $\lambda_{x_i} = \lambda_{x_i^{-1}} = \lambda$. Likewise, the gauge theory defined in Eq. (4.13) is invariant under discrete rotations and reflections when

$$U_{hg,g} \to U_{R_{ij}[hg],R_{ij}[g]}\ , \qquad \text{or} \qquad U_{hg,g} \to U_{R_i[hg],R_i[g]} \quad . \tag{4.46}$$

In Eq. (4.46), the direction of the link variable is rotated or reflected.

The group of rotations (and reflections) on twisted d-dimensional group lattices is isomorphic to the full point group of the corresponding hypercubic lattices. In other words, the relations in Eq. (4.43) hold for the twisted-lattice case as well.

5 Conclusions

In this work, we have solved the 4-dimensional twisted group lattice by finding all the irreducible matrix representations. A key step was prime factorization. Given the irreducible representations, propagators for free field theories can be computed and a perturbation series for an interacting theory can be developed.

In addition, we have shown how to define field theories on group lattices. Scalar interactions are introduced by including polynomials in fields of order three or higher in the action. Gauge theories are obtained à la Wilson by using link variables. Likewise, fermions can be introduced. We have shown that, for the $\{N_{ij} = 2\}$ system, staggered-gamma-matrix structure arises naturally in one sector. In summary, the

situation for twisted group lattices is as good as for regular lattices. The difference between the two lattices is that the former involves a non-trivial space-time structure.

As shown in Sect. 4.3, the 4-dimensional twisted group lattice can be viewed as a non-trivial compactification of a ten-dimensional lattice. The lattice is not even locally a product of a 4-dimensional lattice with a 6-dimensional lattice. It is perhaps curious that superstring theory is naturally formulated in ten dimensions. It would be interesting to develop a compactification of a ten-dimensional continuum theory with non-trivial-holonomy aspects similar to the 4-dimensional twisted group lattices. One can adopt the opposite viewpoint: The twisted lattice can be used as a model for discrete ten-dimensional compactification. A continuum limit can be obtained by taking the lengths L_i to infinity and approaching a critical point, as one does for a regular lattice.[5]

One can exploit the Kaluza-Klein analogy as a means of establishing contact with low-energy phenomenology. Indeed, continuum compactification models contain higher-mass Kaluza-Klein modes. These modes are usually heavy and do not effect low-energy physics. Likewise, the 4-dimensional twisted group lattice has sectors with $\{k_{ij} \neq 0\}$, which are the lattice analogs of Kaluza-Klein modes. If these modes are heavy, then the low-energy theory is governed by the $\{k_{ij} = 0\}$ sector. This sector is equivalent to an $\{N_{ij} = 0\}$ model, which is an ordinary 4-dimensional hypercubic lattice. Hence, if the Kaluza-Klein modes are heavy for a field theory on a twisted group lattice, then the low-energy physics should be similar to the corresponding field theory on a regular lattice. We have checked this idea for a free scalar field on the $\{N_{ij} = 2\}$ lattice. We find that the $\{k_{ij} = 0\}$ sector has the lowest mass. In a continuum limit, the $\{k_{ij} \neq 0\}$ modes become heavy, as expected.

We have also demonstrated that the twisted group lattice possesses discrete translational and rotational invariances. For regular lattices, this is usually sufficient to ensure full Euclidean symmetry in a continuum limit. The same is likely to be true for twisted group lattices, although future work is needed to verify this. In contrast, for a non-trivial quantum plane, defined by the relations $x_j x_i = q_{ij} x_i x_j$, for $i < j$, there are no discrete rotational symmetries analogous to Eq. (4.41), unless the space-time dimension is two or unless $q_{ij} = -1$ for all $i < j$. This is easily checked by explicit computation. It also can be seen by exploiting the magnetic-field analogy [5, 6]. Propagation on a quantum plane is equivalent to a particle of charge e moving on a regular lattice in the presence of field strengths F_{ij} given by $F_{ij} = -i \ln (q_{ij}) / (ea^2)$,

[5] One can also consider taking the N_{ij} to infinity to obtain a continuum limit in the internal space.

where a is the lattice spacing.[6] When such a charged particle moves around a plaquette in the ij plane, its wave function is multiplied by $\exp(iea^2F_{ij}) = q_{ij}$. Since the total magnetic field points in some arbitrary direction, discrete Euclidean rotation invariance is broken. If discrete Euclidean rotation is to be achieved, a more general approach is needed.

From the twisted-group-lattice viewpoint, one can understand the difficulties of defining interacting field theories for quantum hyperplanes. A quantum plane system in the ij direction, with q_{ij} being a rational phase, corresponds to a sector of the free twisted group lattice with k_{ij} determined by $q_{ij} = \exp(2\pi i k_{ij}/N_{ij})$, i. e., Kaluza-Klein modes with internal lattice momenta of $2\pi k_{ij}/N_{ij}$. In a free theory, such modes are decoupled. However, when interactions, which involve the product of three or more fields, are present, those modes couple to each other. One needs to include the other k_{ij} momenta and, hence, other q_{ij} values. For this reason it is probably difficult to define interacting systems for quantum planes. Twisted group lattices are, in some sense, the natural generalization of quantum planes. For twisted group lattices, discrete Euclidean invariances are achieved and interacting field theories are not problematic.

An interesting question is the behavior of non-abelian gauge theories on twisted group lattices. Since non-perturbative effects are expected to be important, one must resort to numerical methods such as Monte Carlo simulations. By viewing the d-dimensional twisted group lattice as a particular $d + d(d-1)/2$ lattice, computer simulations can be performed.

In short, there is much to understand about twisted group lattices, both as a model of discrete compactification and as a possible deformation of space-time structure.

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[6]The field strengths are real only when the q_{ij} are phases, so that the discussion should be restricted to this case.

Appendix: Proof of Irreducibility and Completeness

To prove irreducibility and completeness of the representations, it suffices [14] to show that

$$\sum_{(r)} d_r^2 = o(G) \quad , \tag{A.1}$$

and that

$$\sum_{g \in G} \chi^{(r)}(g)\, \chi^{(s)}\left(g^{-1}\right) = \delta^{(r)(s)} o(G) \quad , \tag{A.2}$$

which expresses the orthogonality of characters. Here, $\chi^{(r)}$ denotes the character of the representation r. The character is obtained by taking the trace of the matrix. Equation (A.1) was already obtained in Eq. (3.37), so that it is only necessary to prove Eq. (A.2).

Express the element g in Eq. (A.2) as

$$g = x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} z_{12}^{n_{12}} z_{13}^{n_{13}} z_{14}^{n_{14}} z_{23}^{n_{23}} z_{24}^{n_{24}} z_{34}^{n_{34}} \quad . \tag{A.3}$$

Recall that a representation is determined by the integers

$$\{k_{12}, k_{13}, k_{14}, k_{23}, k_{24}, k_{34}, k_1, k_2, k_3, k_4\} \quad .$$

Since two representations r and s appear in Eq. (A.2), we use a superscript to distinguish quantities associated with r and s. For example, the representation r corresponds to

$$r \;\leftrightarrow\; \left\{k_{12}^{(r)}, k_{13}^{(r)}, k_{14}^{(r)}, k_{23}^{(r)}, k_{24}^{(r)}, k_{34}^{(r)}, k_1^{(r)}, k_2^{(r)}, k_3^{(r)}, k_4^{(r)}\right\} \quad . \tag{A.4}$$

Using Eqs. (3.1) and (A.3), it follows that

$$\chi^{(r)}(g)\, \chi^{(s)}\left(g^{-1}\right) \;\propto\; \exp\left[2\pi i \sum_{i<j} \frac{\Delta k_{ij} n_{ij}}{N_{ij}}\right] \quad , \tag{A.5}$$

where $\Delta k_{ij} \equiv k_{ij}^{(r)} - k_{ij}^{(s)}$. Hence, the sum over the n_{ij} leads to the conclusion that

$$\sum_{n_{ij}} \chi^{(r)}(g)\, \chi^{(s)}\left(g^{-1}\right) \;=\; 0 \;, \qquad \text{unless } \Delta k_{ij} = 0 \quad . \tag{A.6}$$

For the rest of this appendix, we assume

$$k_{ij}^{(r)} = k_{ij}^{(s)} \;, \qquad \text{for all } i < j \quad . \tag{A.7}$$

When $\Delta k_{ij} = 0$, the sum over n_{ij} produces a factor of

$$\left(\prod_{i<j} \sum_{n_{ij}}\right) 1 \;=\; \prod_{i<j} N_{ij} \quad . \tag{A.8}$$

Given that the k_{ij} are the same for both representations, the following prime and derived quantities are equal

$$\begin{gathered} k'^{(r)}_{ij} = k'^{(s)}_{ij} \;, \qquad N'^{(r)}_{ij} = N'^{(s)}_{ij} \;, \qquad \text{for } 1 \le i < j \le 4 \quad , \\ N'^{(r)}_{\ell} = N'^{(s)}_{\ell} \;, \qquad M^{(r)}_{\ell} = M^{(s)}_{\ell} \quad , \\ a^{(\ell)(r)}_i = a^{(\ell)(s)}_i \;, \quad b^{(\ell)(r)}_i = b^{(\ell)(s)}_i \;, \quad a'^{(\ell)(r)}_i = a'^{(\ell)(s)}_i \;, \quad b'^{(\ell)(r)}_i = b'^{(\ell)(s)}_i \quad , \end{gathered} \tag{A.9}$$

for $i = 1, 2, 3, 4$ and for all ℓ.

Explicit examination of $P_{(N)}$ and $Q_{(N)}$ reveals that the traces $\mathrm{Tr} P^n_{(N)}$ and $\mathrm{Tr} Q^n_{(N)}$ are zero unless $n = 0 \; (\bmod N)$. In prime-factorized form, this holds for each ℓ sector. Hence, for g in the form of Eq. (A.3), $\chi^{(r)}(g) = 0$ unless, for each ℓ,

$$\begin{aligned} \sum_{i=1}^{4} a^{(\ell)}_i n_i &= 0 \pmod{p_\ell^{s_\ell}} \quad , \\ \sum_{i=1}^{4} b^{(\ell)}_i n_i &= 0 \pmod{p_\ell^{s_\ell}} \quad , \\ \sum_{i=1}^{4} a'^{(\ell)}_i n_i &= 0 \pmod{p_\ell^{u_\ell}} \quad , \\ \sum_{i=1}^{4} b'^{(\ell)}_i n_i &= 0 \pmod{p_\ell^{u_\ell}} \quad . \end{aligned} \tag{A.10}$$

The $4L$ constraints in Eq. (A.10) need to be imposed on the n_i in the sum in Eq. (A.2). When these constraints are satisfied,

$$\chi^{(r)}(g)\, \chi^{(s)}\left(g^{-1}\right) \;=\; d_r^2 \exp\left[2\pi i \sum_{i=1}^{4} \frac{\Delta k_i n_i}{L_i}\right] \quad , \tag{A.11}$$

where $\Delta k_i \equiv k^{(r)}_i - k^{(s)}_i$.

Summarizing the situation at this stage,

$$\sum_{g \in G} \chi^{(r)}(g)\, \chi^{(s)}\left(g^{-1}\right) \;=\; d_r^2 \; \delta_{k^{(r)}_{ij}, k^{(s)}_{ij}} \left(\prod_{i<j} N_{ij}\right) \sum_{\substack{n_1, n_2, n_3, n_4 \\ with\ constraints}} \exp\left[2\pi i \sum_{i=1}^{4} \frac{\Delta k_i n_i}{L_i}\right] \;, \tag{A.12}$$

where the constraints on the n_i are given in Eq. (A.10).

A trick can be used to handle the constraints. To save space, it is convenient to define

$$\begin{aligned} a_{1i}^{(\ell)} &\equiv a_i^{(\ell)} \ , \quad a_{2i}^{(\ell)} \equiv b_i^{(\ell)} \ , \quad a_{3i}^{(\ell)} \equiv a_i^{\prime(\ell)} \ , \quad a_{4i}^{(\ell)} \equiv b_i^{\prime(\ell)} \quad , \\ s_{1\ell} &\equiv s_\ell \ , \quad s_{2\ell} \equiv s_\ell \ , \quad s_{3\ell} \equiv u_\ell \ , \quad s_{4\ell} \equiv u_\ell \quad , \end{aligned} \tag{A.13}$$

so that the constraints in Eq. (A.10) can be written in compact form as

$$\sum_{i=1}^{4} a_{ci}^{(\ell)} n_i = 0 \pmod{p_\ell^{s_{c\ell}}} \ , \qquad c = 1,2,3,4 \ , \quad \ell = 1,2,\ldots,L \quad . \tag{A.14}$$

The factor

$$\prod_{\ell=1}^{L}\prod_{c=1}^{4}\left(\frac{1}{p_\ell^{s_{c\ell}}}\sum_{m_c^{(\ell)}=1}^{p_\ell^{s_{c\ell}}}\exp\left[-2\pi i\frac{m_c^{(\ell)}}{p_\ell^{s_{c\ell}}}\sum_{i=1}^{4}a_{ci}^{(\ell)}n_i\right]\right) \tag{A.15}$$

automatically produces the constraints on the n_i when the sums over the $m_c^{(\ell)}$ are performed. In other words,

$$\sum_{\substack{n_1,n_2,n_3,n_4 \\ with\ constraints}} = \left(\prod_{i=1}^{4}\sum_{n_i=1}^{L_i}\right)\left(\prod_{\ell=1}^{L}\prod_{c=1}^{4}\left(\frac{1}{p_\ell^{s_{c\ell}}}\sum_{m_c^{(\ell)}=1}^{p_\ell^{s_{c\ell}}}\exp\left[-2\pi i\frac{m_c^{(\ell)}}{p_\ell^{s_{c\ell}}}\sum_{i=1}^{4}a_{ci}^{(\ell)}n_i\right]\right)\right) \ . \tag{A.16}$$

Hence, we use Eq. (A.16) in Eq. (A.12) and sum over the n_i freely. When the n_i sums are done before the $m_c^{(\ell)}$ sums, one finds a non-zero result if and only if

$$\Delta k_i \ = \ L_i \sum_{\ell,c} \frac{m_c^{(\ell)} a_{ci}^{(\ell)}}{p_\ell^{s_{c\ell}}} \pmod{L_i} \ , \qquad \text{for } i = 1,2,3,4 \quad . \tag{A.17}$$

Using the explicit form of the $a_{ci}^{(\ell)}$ in Eqs. (3.31) and (A.13), one can check that two sets of $m_c^{(\ell)}$ cannot lead to the same Δk_i. This means that, at most, one term in the $m_c^{(\ell)}$ sum contributes. If the momenta $k_i^{(r)}$ and $k_i^{(s)}$ of two representations r and s differ in a way given by Eq. (A.17), then they are related by the element

$$\prod_{\ell=1}^{L}\left(E_{P\times I}^{(\ell)}\right)^{m_1^{(\ell)}}\left(E_{Q^{-1}\times I}^{(\ell)}\right)^{m_2^{(\ell)}}\left(E_{I\times P}^{(\ell)}\right)^{m_3^{(\ell)}}\left(E_{I\times Q^{-1}}^{(\ell)}\right)^{m_4^{(\ell)}} \quad , \tag{A.18}$$

of E, as can be seen from Eqs. (3.35)–(3.38). Consequently,

$$r \sim s \quad , \tag{A.19}$$

if a non-zero result is to be obtained. Summarizing, when the representations r and s are not equivalent, Eq. (A.2) gives zero.

When $r \sim s$, there is a unique non-zero term in the $m_c^{(\ell)}$ sum, and the phase factors in Eqs. (A.12) and (A.16) cancel. Then the sums over the n_i produce $L_1 L_2 L_3 L_4$ and the $1/p_\ell^{s_\ell}$ factors in Eq. (A.16) yield $1/\prod_\ell p_\ell^{2s_\ell+2u_\ell} = 1/d_r^2$. Inserting these results into Eq. (A.12) leads to

$$\begin{aligned}
\sum_{g\in G} \chi^{(r)}(g)\,\chi^{(s)}\left(g^{-1}\right) &= d_r^2\;\delta^{(r)(s)}\frac{L_1 L_2 L_3 L_4}{d_r^2}\left(\prod_{i<j} N_{ij}\right) \\
&= \delta^{(r)(s)}\left(\prod_{i=1}^{4} L_i\right)\left(\prod_{i<j} N_{ij}\right) = \delta^{(r)(s)} o\,(G) \quad . \qquad \text{(A.20)}
\end{aligned}$$

Equations (A.1) and (A.20) guarantee that all the irreducible representations have been found.

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